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Reichenbach's Common Cause Definition on Hilbert Lattices

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The Reichenbachian definition of the common cause is formally generalized for the quantum case in two different ways according to two possible definitions of the conditional probability on a Hilbert lattice, and it is shown that, unlike in the classical case, neither of the two definitions is consistent.

1. INTRODUCTION

Given a probabilistic correlation between two events, this correlation might be explainable in terms of a common cause. Reichenbach defines the notion of common cause (Reichenbach, 1956) and shows that the definition is *consistent* with the explicable correlation, i.e., if two events have a common cause, then they do correlate. We summarize these results briefly in Section 2. In Sections 3 and 4 we generalize the notion of common cause to Hilbert lattices in two different ways according to two different definitions of the conditional probability in the quantum case, and show that the analogue of Reichenbach's theorem does not hold in either case. We give counterexamples when a common cause 'cause' correlation, anticorrelation, and independence, respectively.

2. THE CLASSICAL CASE

Let (i) (Ω, F, p) be a Kolmogorovian probability measure space and let (ii) the conditional probability of E given F be defined as usual by

$$p(E|F) = \frac{p(E \cap F)}{p(F)}$$

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Let $A, B \in \Omega$ be two correlating events, i.e.,

$$p(A \cap B) > p(A)p(B) \tag{1}$$

Reichenbach defines the common cause of the correlation as follows:

Definition 1. An event C is said to be the *common cause* of the correlation between A and B if the events A, B, and C satisfy the following relations:

$$p(A \cap B|C) = p(A|C)p(B|C) \tag{2}$$

$$p(A \cap B|\overline{C}) = p(A|\overline{C})p(B|\overline{C}) \tag{3}$$

$$p(A|C) > p(A|\overline{C}) \tag{4}$$

$$p(B|C) > p(B|\overline{C}) \tag{5}$$

We denote by $p(\cdot|C)$ and $p(\cdot|\overline{C})$ probabilities conditioned on C and non-C, respectively. Now we do not investigate the question under what conditions a common cause satisfying (2)–(5) exists. We rather turn our attention to the question of whether the existence of a common cause really yields correlation. The answer is given by the following:

Theorem 1 (Reichenbach, 1956). Let A, B, and C be elements of a Kolmogorovian probability measure space and let them satisfy (2)–(5). Then A and B correlate, i.e., they satisfy (1).

Proof. In the proof we use the following three equations:

(α) $p(A) = p(C)p(A|C) + p(\overline{C})p(A|\overline{C})$
(β) $p(B) = p(C)p(B|C) + p(\overline{C})p(B|\overline{C})$
(γ) $p(A \cap B) = p(C)p(A|C)p(B|C) + p(\overline{C})p(A|\overline{C})p(B|\overline{C})$

(α) and (β) are identities in a Kolmogorovian probability measure space, (γ) is true if (2)–(3) are true. From these relations we find by some simple computations that

$$p(A \cap B) - p(A)p(B) = p(C)p(\overline{C})[p(A|C) - p(A|\overline{C})][p(B|C) - p(B|\overline{C})]$$

Because of (4)–(5) and under the assumption $0 < p(C) < 1$, we get that $p(A \cap B) - p(A)p(B) > 0$, which was to be proven. ■

So our classical definition is consistent, i.e., the presence of a common cause leads to correlation. But let us go over to the quantum case!

3. FIRST GENERALIZATION

Let (i) $P(H)$ be a Hilbert lattice and W be a pure state represented by the unit vector w. For the projections E and F in the lattice let (ii) the conditional probability of E given F in a state W be defined in the following way:

$$p_w(E|F) = \frac{p_w(E \wedge F)}{p_w(F)} = \frac{\text{Tr}(W(E \wedge F))}{\text{Tr}(WF)}$$

(Now we disregard the logical and mathematical difficulties arising from this generalization of the Bayes rule.) Let $A, B \in P(H)$ and assume a correlation between A and B in the state W, i.e.,

$$p_w(A \wedge B) > p_w(A)p_w(B) \tag{6}$$

We define now the common cause of the correlation in the quantum case:

Definition 2. An event C is said to be the *common cause* of the correlation between A and B if the events A, B, and C satisfy the following relations:

$$p_w(A \wedge B|C) = p_w(A|C)p_w(B|C) \tag{7}$$

$$p_w(A \wedge B|C^\perp) = p_w(A|C^\perp)p_w(B|C^\perp) \tag{8}$$

$$p_w(A|C) > p_w(A|C^\perp) \tag{9}$$

$$p_w(B|C) > p_w(B|C^\perp) \tag{10}$$

Now we show that the analogue of Reichenbach's theorem does not hold in this case. So we claim the following:

Theorem 2. Let A, B, and C be elements of a Hilbert lattice and let them satisfy (7)–(10). Then A and B can either correlate, i.e., $p_w(A \cap B) > p_w(A)p_w(B)$, or anticorrelate, i.e., $p_w(A \cap B) < p_w(A)p_w(\text{B})$; or be independent, i.e., $p_w(A \cap B) = p_w(Ap_w(B)$.

Proof. Let $P(H_3)$ be the projection lattice of the three-dimensional real Hilbert space H_3 with the basis $\{x, y, z\}$ (see Fig. 1). Let $RanC$ be the plane xy, $RanC^\perp$ be the axis z, $RanA$ and $RanB$ be two planes intersecting each other in line x, both having an angle α with z. Let w be in the plane xz meeting with z at an angle β.

We claim that for all $\alpha, \beta \in (0, \pi/2)$, (7)–(10) are satisfied. The conditional probabilities are the following:

$$p_w(A|C) = \frac{p_w(A \wedge C)}{p_w(C)} = \frac{\text{Tr}(W(A \wedge C))}{\text{Tr}(WC)} = \frac{\cos^2\beta}{\cos^2\beta} = 1$$

$$p_w(B|C) = \frac{p_w(B \wedge C)}{p_w(C)} = \frac{\text{Tr}(W(B \wedge C))}{\text{Tr}(WC)} = \frac{\cos^2\beta}{\cos^2\beta} = 1$$

$$p_w(A \wedge B|C) = \frac{p_w(A \wedge B \wedge C)}{p_w(C)} = \frac{\text{Tr}(W(A \wedge B \wedge C))}{\text{Tr}(WC)} = \frac{\cos^2\beta}{\cos^2\beta} = 1$$

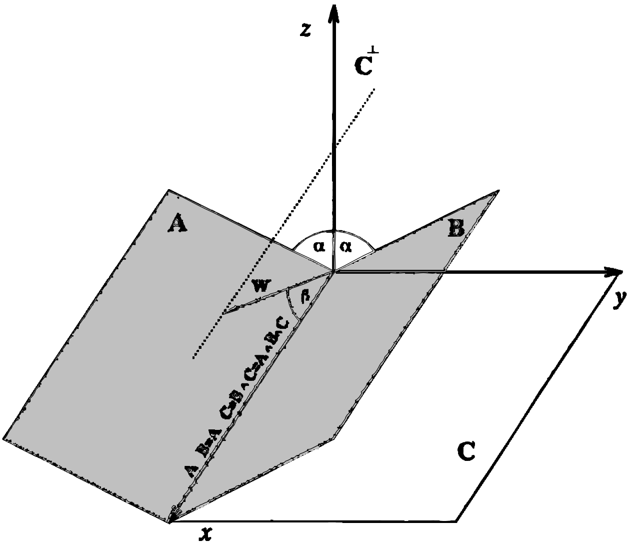

Fig. 1. The projections A, B, and C in $P(H_3)$.

since w is in the plane xz, so its projections onto the plane xy and the axis x are equal, and

$$p_w(A|C^{\perp}) = \frac{p_w(A \wedge C^{\perp})}{p_w(C^{\perp})} = \frac{\mathrm{Tr}(W(A \wedge C^{\perp}))}{\mathrm{Tr}(WC^{\perp})} = 0$$

$$p_w(B|C^{\perp}) = \frac{p_w(B \wedge C^{\perp})}{p_w(C^{\perp})} = \frac{\mathrm{Tr}(W(B \wedge C^{\perp}))}{\mathrm{Tr}(WC^{\perp})} = 0$$

$$p_w(A \wedge B|C^{\perp}) = \frac{p_w(A \wedge B \wedge C^{\perp})}{p_w(C^{\perp})} = \frac{\mathrm{Tr}(W(A \wedge B \wedge C^{\perp}))}{\mathrm{Tr}(WC^{\perp})} = 0$$

since the intersections of A, B, and $A \wedge B$ with $C^{\perp}$ are 0-projections. By these numbers equations (7)–(10) are satisfied:

$$1 = p_w(A \wedge B|C) = p_w(A|C)p_w(B|C) = 1$$

$$0 = p_w(A \wedge B|C^{\perp}) = p_w(A|C^{\perp})p_w(B|C^{\perp}) = 0$$

$$1 = p_w(A|C) > p_w(A|C^{\perp}) = 0$$

$$1 = p_w(B|C) = p_w(B|C^{\perp}) = 0$$

So C can be regarded as the common cause of the correlation between A and B by the above definition.

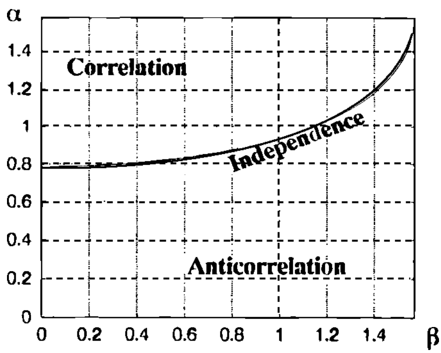

Fig. 2. The parameter space $(\alpha, \beta)_{0,0}^{\pi/2,\pi/2}$.

Let us examine whether there exists a correlation between A and B indeed, i.e., whether (6) is satisfied. The two sides of equation (6) are the following:

$$p_w(A \wedge B) = \mathrm{Tr}(W(A \wedge B)) = \cos^2\beta$$

$$p_w(A)p_w(B) = \mathrm{Tr}(WA)\mathrm{Tr}(WB) = (\cos^2\beta + \sin^2\beta\cos^2\alpha)^2$$

In Fig. 2 we represent the relation between the two sides of (6) in the parameter space $(\alpha, \beta)_{0,0}^{\pi/2,\ \pi/2}$. We can see that the parameter space is divided into two regions by a curve reaching from the line $(0, \alpha)$ to the point $(\pi/2, \pi/2)$ representing the places where $p_w(A \cap B) = p_w(A)p_w(B)$, i.e., where the events A and B are independent. The region 'under' the curve represents the places where $p_w(A \cap B) < p_w(A)p_w(B)$, i.e., where the events A and B anticorrelate. Finally, the region 'above' the curve represents the correlating places where $p_w(A \cap B) > p_w(A)p_w(B)$.

So we have found an example where for two events A and B a third event C can be chosen which can be regarded as the common cause, but A and B do not necessarily correlate; they can anticorrelate or be independent. ∎

In the next section we take another definition of the common cause on the Hilbert lattice using another definition of the conditional probability and examine the validity of the analogue of Reichenbach's theorem.

4. SECOND GENERALIZATION

Let (i) $P(H)$ be a Hilbert lattice and W be a pure state determined by the unit vector w. For the projections E and F in the lattice let (ii) the conditional probability of E given F in a state W be defined in the following way:

$$p_w(E|F) = \frac{\mathrm{Tr}(FWFE)}{\mathrm{Tr}(FWF)}$$

The motivation of this definition comes from the theory of measurement. If we carry out a measurement of an observable represented by the projection F in a pure state W, then the state transforms as follows:

$$W \mapsto \frac{FWF}{\mathrm{Tr}(FWF)}$$

It can be easily seen that the new state is pure again. Let us introduce the following notation for the new pure state: $W_F \equiv EWF/\mathrm{Tr}(FWF)$. The $W \mapsto W_F$ transformation can be regarded as the 'renormalized projection' of the state W onto the subspace *RanF.* This rule is due to Lüders (1951; Bub 1979). Using the above notation, now we are able to define the common cause in terms of this new conditional probability: Let $A, B \in P(H)$ and let there be a correlation between A and B in the state W, i.e.,

$$p_w(A \wedge B) > p_w(A)p_w(B) \tag{11}$$

Definition 3. An event C is said to be the *common cause* of the correlation between A and B if the events A, B, and C satisfy the following relations:

$$\mathrm{Tr}(W_C(A \wedge B)) = \mathrm{Tr}(W_C A)\mathrm{Tr}(W_C B) \tag{12}$$

$$\mathrm{Tr}(W_{C^\perp}(A \wedge B)) = \mathrm{Tr}(W_{C^\perp} A)\mathrm{Tr}(W_{C^\perp} B) \tag{13}$$

$$\mathrm{Tr}(W_C A) > \mathrm{Tr}(W_{C^\perp} A) \tag{14}$$

$$\mathrm{Tr}(W_C B) > \mathrm{Tr}(W_{C^\perp} B) \tag{15}$$

Now we ask again whether A and B correlate, provided there exists a third event C such that conditions (12)–(15) hold. The answer is again negative.

Theorem 3. Let A, B, and C be elements of a Hilbert lattice and let them satisfy (12)–(15). Then A and B do not necessarily correlate.

Proof. In the proof we give a rather technical counterexample which satisfies (12)–(15), but does not satisfy (11). Let us take the same three-dimensional Hilbert lattice $P(H_3)$ as before with the basis $\{x, y, z\}$ (see Fig. 3). Since in equations (12)–(15) C and C^1 do not appear explicitly, in the first step we do not determine these projections; instead we search for two unit perpendicular vectors w_c and $w_{C^\perp}$ which satisfy (12)–(15), and at the end we return to the projections. Let *RanA* and *RanB* be two planes satisfy

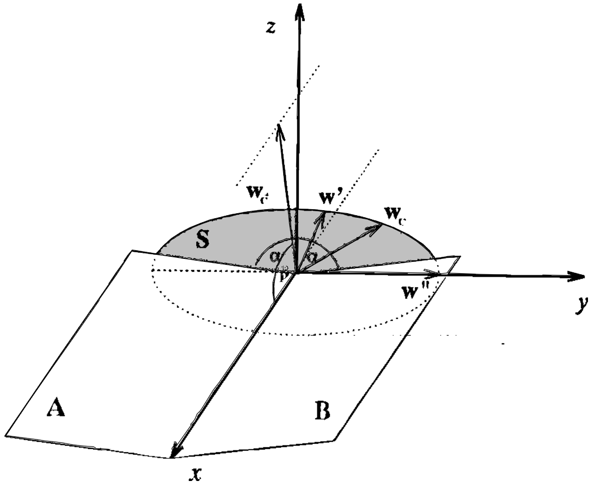

Fig. 3. The position of w_C and $w_C\perp$ in $P(H_3)$.

(12)–(15), and at the end we return to the projections. Let *RanA* and *RanB* be two planes intersecting each other in x meeting with z at an angle α. By Fig. 2 there uniquely exists a vector v in the plane xz for which p_v $(A \wedge B) = p_v(A)p_v(B)$. Let this vector v be $w_C\perp$, so (13) is satisfied.

Our task is now to find a vector w_C perpendicular to $w_C\perp$ so that (12) and (14)–(15) are satisfied. The last two inequalities can be satisfied as follows: Let α tend to $\pi/2$, i.e., let *RanA* and *RanB* tend to the plane xy. Then by Fig. 2, β also tends to $\pi/2$, i.e., $w_C\perp$ tends to the axis z. Let us denote the plane perpendicular to $w_C\perp$ by S. Now this plane is infinitesimally close to the plane xy and to *RanA* and *RanB*. From all this it these follows that for arbitrarily small ε_1 and ε_2 we can choose a δ so that for any α for which $|\pi/2 - \alpha| < \delta$, $p_{w_C}{}^{\perp}(A) = p_{w_C}{}^{\perp}(B) < \varepsilon_1(\delta)$, and for every vector u in the plane S, $p_u(A) > 1 - \varepsilon_2(\delta)$, $p_u(B) > 1 - \varepsilon_2(\delta)$. So (14)–(15) are satisfied for every u in S.

Now let's pick out the vector from the plane S which satisfies also (12). Instead of searching for a vector w_C satisfying p_{w_C} $(A \wedge B) = p_{w_C}$ $(A)p_{w_C}$ (B), we pick out two other vectors w' and w'' for which inequalities hold with the opposite sign, i.e., $p_{w'}$ $(A \wedge B) > p_{w'}$ (A) $p_{w'}$ (B) and $p_{w''}$ $(A \wedge B) < p_{w''}$ (A) $p_{w''}$ (B). Let w' be the vector determined by the intersection of the planes xz and S. In Fig. 2 we can see that w' is in the correlating region, so for w', $p_{w'}$ $(A \wedge B) > p_{w'}$ (B). Let the other vector w'' be determined by the intersection of the planes yz and S which is the axis y itself. For w'', $p_{w''}$ $(A \wedge B) = 0$, since $w'' \perp A \wedge B$, but $p_{w'}$ $(A) \neq 0$ and $p_{w'}$ $(B) \neq 0$, so $p_{w''}$ $(A \wedge B) < p_{w''}$ (B). Now let us use the continuity of the $p_u(\cdot)$ function on the plane S. If there is a vector w' for which $p_{w'}(A \wedge B) > p_{w'}(A)$

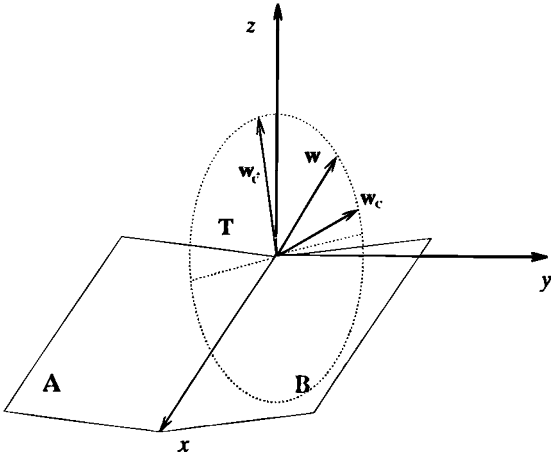

Fig. 4. The position of w in $P(H_3)$.

$p_{w'}(B)$ and a vector w'' for which $p_{w''}(A \wedge B) < p_{w''}(A)\, p_{w''}(B)$, then there must be a vector between them in the plane S for which $p_w(A \wedge B) = p_w(A)p_w(B)$. Let this vector be w_C, so (12) is fulfilled.

So we have found two vectors w_C and $w_{C\perp}$ for which (12)–(15) are satisfied. What are the projections C and $C^\perp$, and what is the original w vector? Let C be the projection for which *RanC* is the plane S, and let $C^\perp$ be the projection determined by $w_C{}^\perp$. Then w can be any of the vectors in the plane T spanned by w_C and $w_C{}^\perp$ except for w' and w''.

Now let us choose a possible w for which independence or anticorrelation happens. Let w be the vector determined by the intersection of the planes yz and T (see Fig. 4). For w, $p_w(A \wedge B) = 0$, since w is in the plane yz. Now there are two possibilities: In the case that $p_w(B) = 0$ or $p_w(A) = 0$, then $p_w(A \wedge B) = p_w(A)p_w(B)$, i.e., A and B are independent; in the case that $p_w(B) \neq 0$ and $p_w(A) \neq 0$, then $p_w(A \wedge B) < p_w(A)p_w(B)$, i.e., A and B anticorrelate. So our counter example satisfies (12)–(15), but not (11), and this was to be proven. ■

So the consistency does not hold for either of the two definitions of the common cause in the quantum case.

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